Entanglement and Classical Correlations in the Quantum Frame

E. G. Beltrametti[1,2,5] and S. Bugajski[3,4]

The frame of classical probability theory can be generalized by enlarging the usual family of random variables in order to encompass nondeterministic ones. This leads to a frame in which two kinds of correlations emerge: the classical correlation that is coded in the mixed state of the physical system and a new correlation, to be called probabilistic entanglement, which may occur also at pure states. We examine to what extent this characterization of correlations can be applied to quantum mechanics. Explicit calculations on simple examples outline that a same quantum state can show only classical correlations or only entanglement depending on its statistical content; situations may also arise in which the two kinds of correlations compensate each other.

KEY WORDS: quantum mechanics; correlations; entanglement; mixed states.

1. INTRODUCTION

The standard framework of classical statistical mechanics makes use of a convex set of states having the structure of a simplex, and adopts a family of observables, or random variables, which have a deterministic nature. More specifically, the states form the set $M_1^+(\Omega)$ of the probability measures on a measurable space Ω whose points—hence the Dirac measures on Ω to be denoted δ_ω, $\omega \in \Omega$—represent the pure states. An observable taking values in a measurable space Ξ corresponds to an affine map

$$A : M_1^+(\Omega) \to M_1^+(\Xi),$$

and the deterministic requirement is mirrored by the condition that A has no dispersion on pure states, namely Dirac measures are mapped into Dirac measures,

[1] Department of Physics, University of Genoa, Genoa, Italy.

[2] Instituto Nazionale di Fisica Nucleare, Sezione di Genova, Genoa, Italy.

[3] Institute of Physics, University of Silesia, Katowice, Poland.

[4] Institute of Pure and Applied Informatics, Polish Academy of Sciences, Katowice, Poland.

[5] To whom correspondence should be addressed at Department of Physics, University of Genoa, Genoa, Italy; e-mail: beltrametti@genova.infin.it.

so that the observable A becomes represented by a measurable function $\Omega \rightarrow \Xi$. Any two observables have a unique joint observable and a correlation between their outcomes can occur only at mixed states.

If the above framework is generalized by dropping out the deterministic requirement, so allowing also observables that have dispersion on pure states, we get a frame that has been discussed in Beltrametti and Bugajski (1995, 1996) and Bugajski (1996, 2001): the set of states is still the simplex $M_1^+(\Omega)$ but now the observables need not map Dirac measures on Ω into Dirac measures on the pertaining outcome space. Any two observables admit a joint observable but the latter is nonunique when the two observables have an indeterministic nature: a correlation between their outcomes is now relative to the considered joint observable and it need not vanish at pure states. In Beltrametti and Bugajski (2002) it is shown that we can separate two kinds of correlation: the *classical correlation*, which occurs only at a mixed state and is coded in the way the pure states are mixed up to get the nonpure state in question, and the *probabilistic entanglement* generated by the joint observable considered, and occurring also in pure states. Both correlations can be exhaustively characterized by corresponding density functions (correlation functions). As the name suggests, the probabilistic entanglement is analogous to the corresponding quantum concept.

In Section 2 we examine to what extent the notions of classical correlation and of probabilistic entanglement can be transferred to the quantum context: we will point at the fact the nonsimplex structure of the set of quantum states gives rise to ambiguities in separating classical correlation and entanglement when mixed states are considered. A quantum state can always be decomposed into a (possibly trivial) convex combination of pure states, so that it admits a representation in the set $M_1^+(\Omega_{\mathcal{H}})$ of the probability measures on the measurable space $\Omega_{\mathcal{H}}$ of the one-dimensional projectors of the Hilbert space $\mathcal{H}$, but this representation is in general nonunique. In fact, the observables adopted by quantum mechanics do not separate $M_1^+(\Omega_{\mathcal{H}})$: they define a partition of $M_1^+(\Omega_{\mathcal{H}})$ into equivalence classes that correspond to the density operators of $\mathcal{H}$. Recalling that mixed quantum states are associated with nontrivial density operators of $\mathcal{H}$ and that pure states are associated with one-dimensional projectors, it is indeed well known that the convex decomposition of a density operator into one-dimensional projectors is nonunique. In Section 3 we will discuss a simple example that emphasizes the ambiguities said above.

The problem of defining the notion of entanglement and of classical correlation at mixed quantum states, and the related issue of characterizing the states that can (or cannot) exhibit entanglement, have received attention in the literature under different perspectives: as actual guides to the vast literature could serve Horodecki *et al.* (2001) and Keyl (2002); we also mention Henderson and Vedral (2001) and the rigorous approach of Majewski (2002) where some ideas similar to ours were formulated.

Let us recall that a quantum observable taking values in the measurable space Ξ can be represented by a POV-measure $E : \mathcal{B}(\Xi) \to \mathcal{L}(\mathcal{H})$, where $\mathcal{B}(\Xi)$ is σ-Boolean algebra of subsets of Ξ and $\mathcal{L}(\mathcal{H})$ is the family of the positive operators of $\mathcal{H}$ (see, e.g., Busch *et al.*, 1995, 1996). If Ξ is the set of the reals then $\mathcal{B}(\Xi)$ is typically the Boolean algebra of the Borel subsets; if Ξ is a finite set then $\mathcal{B}(\Xi)$ is typically the Boolean algebra of all subsets of Ξ.

Writing $\mathcal{S}(\mathcal{H})$ for the convex set of the density operators of $\mathcal{H}$, an equivalent representation of an observable taking values in Ξ is given by an affine map A of $\mathcal{S}(\mathcal{H})$ into the set $M_1^+(\Xi)$ of all probability measures on Ξ. For a given quantum state $D \in \mathcal{S}(\mathcal{H})$ the measure $A(D)$ is the outcome measure that is the result of a measurement of the observable A on the state D. We write E^A to denote the POV-measure associated to A and recall that the probability of getting a value of the observable A in the set $X \in \mathcal{B}(\Xi)$ at the state D is given by the basic quantum rule

$$A(D)(X) = \mathrm{Tr}(E^A(X)D). \tag{1}$$

If the POV-measure E^A reduces to a PV-measure on the real line, then the observable A is known to correspond to a self-adjoint operator of $\mathcal{H}$.

Notice that when Ξ has the form of a Cartesian product $\Xi_1 \times \Xi_2$ an observable $A : S(\mathcal{H}) \to M_1^+(\Xi_1 \times \Xi_2)$ defines two observables, $A_i : S(\mathcal{H}) \to M_1^+(\Xi_i)$, $i = 1, 2$, by $A_i(D) : \prod_i(A(D))$ where $\prod_i : M_1^+(\Xi_1 \times \Xi_2) \to M_1^+(\Xi_i)$ is the marginal projection. The observable A is then said to be a quantum joint observable of A_1 and A_2. However, for a pair of observables $A_i : S(\mathcal{H}) \to M_1^+(\Xi_i)$, $i = 1, 2$, the existence of a quantum joint observable is not ensured.

2. CORRELATIONS

In probability theory a correlation between two parameter sets Ξ_1 and Ξ_2 is understood as a particular property of a probability measure ν on $\Xi_1 \times \Xi_2$, namely

$$\nu \neq \nu_1 \boxtimes \nu_2,$$

where $\nu_i = \prod_i \nu$, $i = 1, 2$, is the marginal measure on Ξ_i, and $\boxtimes$ stands for the product of measures. Thus, the notion of correlation just corresponds to the lack of independence (to a "mutual relationship," according to the Oxford Advanced Learner's Dictionary).

Consequently, we can say that a correlation between Ξ_1 and Ξ_2 encoded in $\nu \in M_1^+(\Xi_1 \times \Xi_2)$ is what distinguishes ν from $\nu_1 \boxtimes \nu_2$. If we want to find a formal characterization of such a correlation we have to find how to describe the "difference" between ν and $\nu_1 \boxtimes \nu_2$. An exhaustive description of this "difference" is provided by the density function (the Radon–Nikodym derivative, see e.g., Bauer, 1981; Billingsley, 1979) of ν w.r.t. $\nu_1 \boxtimes \nu_2$. Consequently, everything one can say about a correlation between Ξ_1 and Ξ_2 encoded in $\nu \in M_1^+(\Xi_1 \times \Xi_2)$ is contained

in the density function

$$\rho := \frac{d\nu}{d(\nu_1 \boxtimes \nu_2)},$$

which is a real-valued positive function on $\Xi_1 \times \Xi_2$. The existence of this density function is ensured whenever ν is a discrete measure (Beltrametti and Bugajski, 2002). If in particular Ξ_1 and Ξ_2 and finite sets, the relationship between ν and $\nu_1 \boxtimes \nu_2$ will take the form

$$\nu(X) = \sum_{(\xi_1, \xi_2) \in X} \rho \cdot \nu \boxtimes \nu_2(\xi_1, \xi_2), \quad \xi_1 \in \Xi_1, \xi_2 \in \Xi_2$$

for every $X \subseteq \Xi_1 \times \Xi_2$. In this case ρ can be simply calculated by pointwise dividing the two measures ν and $\nu_1 \boxtimes \nu_2$:

$$\rho(\xi_1 \xi_2) = \frac{\nu(\xi_1, \xi_2)}{\nu_1 \boxtimes \nu_2(\xi_1, \xi_2)}. \tag{2}$$

The presence of a correlation is mirrored by the fact that ρ is not the constant unit function.

We will be interested in the case in which the two correlated sets Ξ_1 and Ξ_2 are value sets (sets of outcomes) of two observables, while the probability measure ν on $\Xi_1 \times \Xi_2$ is the result of the measurement of a joint observable of them.

In the standard context of the classical statistical mechanics, where the set of states is the simplex $M_1^+(\Omega)$ and only deterministic observables come into play, any two observables $A_1 : M_1^+(\Omega) \to M_1^+(\Xi_1)$, $A_2 : M_1^+(\Omega) \to M_1^+(\Xi_2)$ always admit the unique joint observable $A_1 \boxtimes A_2$ defined by its action on the pure states

$$A_1 \boxtimes A_2(\delta_\omega) := A_1(\delta_\omega) \boxtimes A_2(\delta_\omega) \quad \text{for every } \omega \in \Omega, \tag{3}$$

and extended by affinity to the whole $M_1^+(\Omega)$. Thus, when we speak of a correlation between A_1 and A_2 at a state $\mu \in M_1^+(\Omega)$ the reference to the joint observable $A_1 \boxtimes A_2$ is compulsory, and we have just to compare the two measures $A_1 \boxtimes A_2(\mu)$ and $A_1(\mu) \boxtimes A_2(\mu)$. What we get is the classical correlation characterized by the density function (the Radon–Nicodym derivative)

$$\rho_c := \frac{d(A_1 \boxtimes A_2(\mu))}{d(A_1(\mu) \boxtimes A_2(\mu))}. \tag{4}$$

In view of Eq. (3) $\rho_c = 1$ at pure states: a nontrivial classical correlation can appear only at mixed states.

If we go to the generalization of the standard classical frame by allowing also indeterministic observables, then the unicity of the joint observable breaks down: besides $A_1 \boxtimes A_2$ other joint observables become possible (see Beltrametti and Bugajski, 1996, 2002; Bugajski, 1996). When we speak of a correlation between A_1 and A_2 at a state $\mu \in M_1^+(\Omega)$ we have now to specify which joint observable $J(A_1, A_2)$ we refer to and we are naturally led to compare the two measures

$J(A_1, A_2)(\mu)$ and $A_1(\mu) \boxtimes A_2(\mu)$, thus getting the correlation characterized by the density function

$$\rho_t := \frac{d(J(A_1, A_2)(\mu))}{d(A_1(\mu) \boxtimes A_2(\mu))}. \tag{5}$$

As discussed in Beltrametti and Bugajski (2002) this correlation can be, in general, partitioned into two parts by first comparing the measure $J(A_1, A_2)(\mu)$ with $A_1 \boxtimes A_2(\mu)$ and then comparing the measure $A_1 \boxtimes A_2(\mu)$ with $A_1(\mu) \boxtimes A_2(\mu)$. The second step provides just the classical correlation said above, while the first step provides a correlation to be called *entanglement*. Clearly, the entanglement can emerge only when the joint observable referred to differs from $A_1 \boxtimes A_2$, namely from the classical joint observable. The density function associated to the entanglement will then be

$$\rho_e := \frac{d(J(A_1, A_2)(\mu))}{d(A_1 \boxtimes A_2(\mu))}, \tag{6}$$

and known properties of the Radon–Nicodym derivatives (see, e.g., Bauer, 1981 Corollary 2.9.4, or Billingsley, 1979, Sect. 32) give the product rule

$$\rho_t = \rho_c \cdot \rho_e. \tag{7}$$

This motivates for ρ_t the name of total correlation (hence the notation).

Let us now come to the quantum frame, and consider two quantum observables $A_i : S(\mathcal{H}) \to M_1^+(\Xi_i)$, $i = 1, 2$, admitting a joint observable: this is the case, for instance, when one deals with real valued observables represented by commuting self-adjoint operators (the joint observable is then unique). We can say that the two observables are correlated at the quantum state $D \in S(\mathcal{H})$, relative to the given quantum joint observable $J(A_1, A_2) : S(\mathcal{H}) \to M_1^+(\Xi_1 \times \Xi_2)$ iff

$$J(A_1, A_2)(D) \neq A_1(D) \boxtimes A_2(D).$$

The total correlation between the quantum observables A_1 and A_2 relative to the joint observable A (at the state D) is then exhaustively described, as in Eq. (5), by the total correlation function

$$\rho_t = \frac{d(J(A_1, A_2)(D))}{d(A_1(D) \boxtimes A_2(D))}. \tag{8}$$

But if we tackle the problem of separating the classical correlation and the entanglement then we are faced with the translation to the quantum frame of the product $A_1 \boxtimes A_2$. We can mirror Eq. (3) by defining $A_1 \boxtimes A_2$ on the pure states according to

$$A_1 \boxtimes A_2(P) := A_1(P) \boxtimes A_2(P) \tag{9}$$

for every one-dimensional projector P of $\mathcal{H}$. But the extension by affinity to the whole set of quantum states $S(\mathcal{H})$ makes sense only if we refer to a specific convex decomposition into pure states of the mixed state D, and this decomposition is

known to be nonunique. In other words, if the mixed state (density operator) D under discussion admits the convex decomposition into pure states

$$D = \sum_i w_i P_i, \tag{10}$$

where the w_i's are positive numbers whose sum is 1 and the P_i's are one-dimensional projectors, then we can affinely define $A_1 \boxtimes A_2$ on the r.h.s. of Eq. (10) getting the measure

$$\sum_i w_i A_1 \boxtimes A_2(P_i) = \sum_i w_i A_1(P_i) \boxtimes A_2(P_i), \tag{11}$$

but this measure is not invariant under different choices of the convex decomposition of D.

In view of the above fact one can speak of classical correlation and of entanglement in the quantum context only with reference to a given convex decomposition of the (mixed) state under discussion. The corresponding density functions will read (see Eqs. (4) and (6))

$$\rho_c := \frac{d\left(\sum_i w_i A_1(P_i) \boxtimes A_2(P_i)\right)}{d(A_1(D) \boxtimes A_2(D))} \tag{12}$$

and

$$\rho_e := \frac{d(J(A_1 A_2)(D))}{d\left(\sum_i w_i A_1(P_i) \boxtimes A_2(P_i)\right)}. \tag{13}$$

Let us stress that only the product $\rho_c \cdot \rho_e$ that equals ρ_t (see Eq. (7)) has the property of being invariant under different convex decompositions of the quantum state D, while neither ρ_c nor ρ_e have such an invariance.

3. A TWO-QUBIT EXAMPLE

We will illustrate the introduced concepts on a simple quantum-mechanical example based on a Hilbert space of the form $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$: it can be viewed as the composition of two spin-1/2 or as a two-qubit system.

Let $\{\psi_+, \psi_-\}$ be an orthonormal basis of $\mathbb{C}^2$ and let P_+, P_- be the corresponding (one-dimensional) projectors. A canonical orthonormal basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$ is provided by $\{\psi_+ \otimes \psi_+, \psi_- \otimes \psi_-, \psi_+ \otimes \psi_-, \psi_- \otimes \psi_+\}$ and the associated one-dimensional projectors read:

$$P_{++} = P_+ \otimes P_+, \quad P_{--} = P_- \otimes P_-, \quad P_{+-} = P_+ \otimes P_-, \quad P_{-+} = P_- \otimes P_+.$$

Consider the two observables $A_i : S(\mathbb{C}_2 \otimes \mathbb{C}_2) \to M_1^+(\{\frac{1}{2}, -\frac{1}{2}\})$, $i = 1, 2$, described by the self-adjoint operators on $\mathbb{C}^2 \otimes \mathbb{C}^2$:

$$\hat{A}_1 \left(\frac{1}{2} P_+ - \frac{1}{2} P_- \right) \otimes I, \quad \hat{A}_2 = I \otimes \left(\frac{1}{2} P_+ - \frac{1}{2} P_- \right),$$

where I denotes the identity operator in C^2. We can view A_i as the observable describing the z-component of the spin of the ith subsystem. According to quantum mechanics the only admissible joint observable of A_1, A_2 is the observable $J(A_1, A_2) : S(C^2 \otimes C^2) \to M_1^+(\{\frac{1}{2}, -\frac{1}{2}\} \times \{\frac{1}{2}, -\frac{1}{2}\})$ which corresponds to the PV measure $E^{J(A_1,A_2)}$ defined by

$$E^{J(A_1,A_2)} \left(\frac{1}{2}, \frac{1}{2} \right) = P_{++}, \quad E^{J(A_1,A_2)} \left(-\frac{1}{2}, -\frac{1}{2} \right) = P_{--},$$

$$E^{J(A_1,A_2)} \left(\frac{1}{2}, -\frac{1}{2} \right) = P_{+-}, \quad E^{J(A_1,A_2)} \left(-\frac{1}{2}, \frac{1}{2} \right) = P_{-+}.$$

We will be concerned with the correlation between A_1 and A_2 relative to the joint observable $J(A_1, A_2)$ at various quantum states: this will point at the fact that, in the quantum frame, the splitting of the correlation into a classical part and an entanglement might become a matter of convention.

In the sequel we will have to compute the various measures involved in the correlations of interest: to do that we will refer to Eq. (1), noticing that the trace is linear and that for a pure state, say P, the r.h.s. of Eq. (1) takes the form $(\phi, E^A(X)\phi)$, where ϕ is any unit vector in the one-dimensional subspace onto which P projects. We will write $\eta_{\frac{1}{2}}$ to denote the Dirac measure on $\{\frac{1}{2}, -\frac{1}{2}\}$ concentrated at the value $\frac{1}{2}$ (similarly for $\eta_{-\frac{1}{2}}$), and $\eta_{(\frac{1}{2}, \frac{1}{2})}$ to denote the Dirac measure on $\{\frac{1}{2}, -\frac{1}{2}\} \times \{\frac{1}{2}, -\frac{1}{2}\}$ concentrated at the point $(\frac{1}{2}, \frac{1}{2})$ (similarly for $\eta_{(\frac{1}{2}, -\frac{1}{2})}$, $\eta_{(-\frac{1}{2}, \frac{1}{2})}$, $\eta_{(-\frac{1}{2}, -\frac{1}{2})}$).

3.1. Separable Mixed State

A *separable mixed state* is represented by a density operator that decomposes into the convex combination of pure product states. A canonical example is provided by the density operator

$$D = w_1 P_{++} + w_2 P_{--} + w_3 P_{+-} + w_4 P_{-+}. \tag{14}$$

In order to get the total correlation we have now to compare the two measures $J(A_1, A_2)(D)$ and $A_1(D) \boxtimes A_2(D)$.

The measure $J(A_1, A_2)(D)$ on $\{\frac{1}{2}, -\frac{1}{2}\} \times \{\frac{1}{2}, -\frac{1}{2}\}$ is easily obtained looking at the explicit expression of $E^{J(A_1,A_2)}$ given above. We get

$$J(A_1, A_2)(D) = w_1 \eta_{(\frac{1}{2}, \frac{1}{2})} + w_2 \eta_{(-\frac{1}{2}, -\frac{1}{2})} + w_3 \eta_{(\frac{1}{2}, -\frac{1}{2})} + w_4 \eta_{(-\frac{1}{2}, \frac{1}{2})}.$$

The measure $A_1(D)$ on $\{\frac{1}{2}, -\frac{1}{2}\}$ can be obtained by an analogous procedure: noticing that $E^{A_1}(\frac{1}{2}) = P_+ \otimes I$ and $E^{A_1}(-\frac{1}{2}) = P_- \otimes I$ we get,

$$A_1(D) = (w_1 + w_3)\eta_{\frac{1}{2}} + (w_2 + w_4)\eta_{-\frac{1}{2}}.$$

Similarly we have

$$A_2(D) = (w_1 + w_4)\eta_{\frac{1}{2}} + (w_2 + w_3)\eta_{-\frac{1}{2}}.$$

Hence the product measure, $A_1(D) \boxtimes A_2(D)$ takes the form

$$A_1(D) \boxtimes A_2(D) = (w_1 + w_3)(w_1 + w_4)\eta_{(\frac{1}{2}, \frac{1}{2})} + (w_2 + w_4)(w_2 + w_3)\eta_{(-\frac{1}{2}, -\frac{1}{2})}$$
$$+ (w_1 + w_3)(w_2 + w_3)\eta_{(\frac{1}{2}, -\frac{1}{2})} + (w_2 + w_4)(w_1 + w_4)\eta_{(-\frac{1}{2}, \frac{1}{2})}.$$

The density function of the total correlation between A_1 and A_2 at the state D will then turn out to be (see Eqs. (2) and (8)):

$$\rho_t\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{w_1}{(w_1 + w_3)(w_1 + w_4)}, \quad \rho_t\left(-\frac{1}{2}, -\frac{1}{2}\right) = \frac{w_2}{(w_2 + w_4)(w_2 + w_3)},$$

$$\rho_t\left(\frac{1}{2}, -\frac{1}{2}\right) = \frac{w_3}{(w_1 + w_3)(w_2 + w_3)}, \quad \rho_t\left(-\frac{1}{2}, \frac{1}{2}\right) = \frac{w_4}{(w_2 + w_4)(w_1 + w_4)}.$$

In order to examine how this total correlation splits into classical correlation and entanglement we have now to evaluate the measure (see Eqs. (9) and (11))

$$w_1 A_1 \boxtimes A_2(P_{++}) + w_2 A_1 \boxtimes A_2(P_{--}) + w_3 A_1 \boxtimes A_2(P_{+-}) + w_4 A_1 \boxtimes A_2(P_{-+}),$$

where $A_1 \boxtimes A_2(P_{++}) = A_1(P_{++}) \boxtimes A_2(P_{++})$ and so on. The calculation goes as before: for instance we have $A_1(P_{++}) = A_2(P_{++}) = \eta_{\frac{1}{2}}$ so that $A_1(P_{++}) \boxtimes A_2(P_{++}) = \eta_{(\frac{1}{2}, \frac{1}{2})}$, and similarly for the other terms. The result is that the measure above equals exactly the measure $J(A_1, A_2)(D)$ said before. This means that the density function ρ_c of the classical correlation coincides with ρ_t (see Eq. (12)) while the density function ρ_e (see Eq. (13)) is the constant unit function. In other words, the total correlation between A_1 and A_2 at the state D appears to be entirely a classical correlation, without any entanglement coming into play. In the Appendix we will prove that the absence of entanglement holds true also for every bipartite separable mixed state.

3.2. Bell Diagonal State

Instead of the canonical basis $\{\psi_+ \otimes \psi_+, \psi_- \otimes \psi_-, \psi_+ \otimes \psi_-, \psi_- \otimes \psi_+\}$ used before, let us now turn to the Bell basis

$$\Phi_1 := \frac{1}{\sqrt{2}}(\psi_+ \otimes \psi_+ + \psi_- \otimes \psi_-), \quad \Phi_2 := \frac{1}{\sqrt{2}}(\psi_+ \otimes \psi_+ - \psi_- \otimes \psi_-),$$

$$\Phi_3 := \frac{1}{\sqrt{2}}(\psi_+ \otimes \psi_- + \psi_- \otimes \psi_+), \quad \Phi_4 := \frac{1}{\sqrt{2}}(\psi_+ \otimes \psi_- - \psi_- \otimes \psi_+),$$

and let P_1, P_2, P_3, P_4 be the corresponding one-dimensional projectors.

A convex combination of the form

$$D' = w_1' P_1 + w_2' P_2 + w_3' P_3 + w_4' P_4 \tag{15}$$

is called a *Bell diagonal state* (Bennet *et al.*, 1996). We are going to obtain the correlation functions for the observables A_1, A_2 at such a state.

The outcome measure $J(A_1, A_2)(D')$ of the quantum joint observable of A_1 and A_2 is easily found to be

$$J(A_1, A_2)(D') = \frac{1}{2}(w_1' + w_2')\left(\eta_{(\frac{1}{2}, \frac{1}{2})} + \eta_{(-\frac{1}{2}, -\frac{1}{2})}\right)$$
$$+ \frac{1}{2}(w_3' + w_4')\left(\eta_{(\frac{1}{2}, -\frac{1}{2})} + \eta_{(-\frac{1}{2}, \frac{1}{2})}\right).$$

The two measures $A_1(D')$ and $A_2(D')$ have the uniform structure $\frac{1}{2}\eta_{\frac{1}{2}} + \frac{1}{2}\eta_{-\frac{1}{2}}$ so that also their product is uniformly distributed over the four-point space $\{\frac{1}{2}, -\frac{1}{2}\} \times \{\frac{1}{2}, -\frac{1}{2}\}$:

$$A_1(D') \boxtimes A_2(D') = \frac{1}{4}\left(\eta_{(\frac{1}{2}, \frac{1}{2})} + \eta_{(-\frac{1}{2}, -\frac{1}{2})} + \eta_{(\frac{1}{2}, -\frac{1}{2})} + \eta_{(-\frac{1}{2}, \frac{1}{2})}\right).$$

Therefore, the density function of the total correlation between A_1 and A_2 at the Bell state D' is

$$\rho_t'\left(\frac{1}{2}, \frac{1}{2}\right) = \rho_t'\left(-\frac{1}{2}, -\frac{1}{2}\right) = 2(w_1' + w_2'),$$
$$\rho_t'\left(\frac{1}{2}, -\frac{1}{2}\right) = \rho_t'\left(-\frac{1}{2}, \frac{1}{2}\right) = 2(w_3' + w_4').$$

In order to see how this total correlation could be separated into classical correlation and entanglement we must go to the measure (see Eq. (11))

$$w_1' A_1 \boxtimes A_2(P_1) + w_2' A_1 \boxtimes A_2(P_2) + w_3' A_1 \boxtimes A_2(P_3) + w_4' A_1 \boxtimes A_2(P_4),$$

which is easily seen to coincide with the uniform product measure $A_1(D') \boxtimes A_2(D')$. Therefore, by inspection of Eqs. (12) and (13), we conclude that $\rho_e' = \rho_t'$ while ρ_c' is the constant unit function. In other words, the correlation between A_1 and A_2 at the Bell state D' appears to be entirely an entanglement without any classical correlation coming into play.

The absence of any classical correlation that we have found seems to disagree with the result of Henderson and Vedral (2001), where a numerical measure for classical correlation is introduced, which does not vanish at some Bell diagonal state. This might point at the fact that such a numerical measure does not fully capture our notion of classical correlation.

A somewhat similar disagreement with previous literature occurs also when we look at the entanglement density function $\rho_e'(= \rho_t')$ given above which is nonconstant whenever $w_1' + w_2' \neq w_3' + w_4'$. Indeed, according to Bennet *et al.* (1996) and Keyl (2002) a Bell diagonal state shows entanglement only if one of the weights w_1', w_2', w_3', w_4' (in our notations) is bigger than $\frac{1}{2}$: clearly, this would

imply the inequality $w_1' + w_2' \neq w_3' + w_4'$ but the reverse implication does not hold. Again, this might point at the fact that the numerical measure of entanglement introduced in Bennet *et al.* (1996) (the "entanglement of formation") does not cover exactly our definition of entanglement.

3.3. A Degenerate State

As far as the density operators D and D' considered in Sections 3.1 and 3.2 have no degenerate eigenvalues, that is as far as the w_i's and the w_i's in Eqs. (14) and (15) are pairwise distinct, it is guaranteed that D and D' represent distinct quantum states. In this case the fact that the correlation between A_1 and A_2 at the state D is purely classical while at the state D' it is just entanglement makes no problem. But a peculiar feature emerges when we consider degenerate eigenvalues, for instance when we assume

$$w_1 = w_2 = w_1' = w_2' = a \quad \text{and} \quad w_3 = w_4 = w_3' = w_4' = b$$

with $a + b = \frac{1}{2}$. In this case D and D' actually represent the same quantum state, say D_d, since the only difference among their convex decompositions is a different choice of an orthonormal basis within the degenerate eigenspaces.

As expected we have now $\rho_t = \rho_t'$ explicitly:

$$\rho_t\left(\frac{1}{2}, \frac{1}{2}\right) = \rho_t'\left(\frac{1}{2}, \frac{1}{2}\right) = \rho_t\left(-\frac{1}{2}, -\frac{1}{2}\right) = \rho_t'\left(-\frac{1}{2}, -\frac{1}{2}\right) = 4a,$$

$$\rho_t\left(\frac{1}{2}, -\frac{1}{2}\right) = \rho_t'\left(\frac{1}{2}, -\frac{1}{2}\right) = \rho_t\left(-\frac{1}{2}, \frac{1}{2}\right) = \rho_t'\left(-\frac{1}{2}, \frac{1}{2}\right) = 4b.$$

But, according to the results of Sections 3.1 and 3.2, we have now that this correlation appears to be entirely a classical correlation if we refer to the convex combination

$$a P_{++} + a P_{--} + b P_{+-} + b P_{-+} \tag{16}$$

while it appears to be entirely an entanglement if we refer to the convex combination

$$a P_1 + a P_2 + b P_3 + b P_4 \tag{17}$$

despite the fact that these two convex combinations correspond to the same quantum state.

This result emphasizes the fact that the separation of entanglement and classical correlation is possible only if we know the *statistical content* of the quantum mixed state, i.e., the actual decomposition of the mixed state into a convex combination of pure states, but this *statistical content* is in general not uniquely specified by the von Neumann description of quantum mixed states.

Let us further remark that also the convex combination

$$a P_{++} + a P_{--} + b P_3 + b P_4 \tag{18}$$

represents the same quantum state D_d expressed by Eq. (16) or by Eq. (17). Clearly, the total corelation between A_1 and A_2 at this new convex combination is the same as before, but now the density functions of the classical correlation and of the entanglement turn out to be

$$\rho_c \left(\frac{1}{2}, \frac{1}{2} \right) = \rho_c \left(-\frac{1}{2}, -\frac{1}{2} \right) = 2(2a + b), \quad \rho_c \left(\frac{1}{2}, -\frac{1}{2} \right) = \rho_c \left(-\frac{1}{2}, \frac{1}{2} \right) = 2b,$$

and

$$\rho_e \left(\frac{1}{2}, \frac{1}{2} \right) = \rho_e \left(-\frac{1}{2}, -\frac{1}{2} \right) = \frac{2a}{2a + b}, \quad \rho_e \left(\frac{1}{2}, -\frac{1}{2} \right) = \rho_e \left(-\frac{1}{2}, \frac{1}{2} \right) = 2.$$

Thus, if one refers to the convex decomposition of Eq. (18), the total correlation appears to be partially a classical correlation and partially an entanglement.

In the totally degenerate case $a = b(= \frac{1}{4})$, hence in the case of the "most mixed" state, the density function of the total correlation is, as expected, the constant unit function, no matter which convex decomposition one refers to. But, if the convex decomposition of Eq. (18) is referred to, there is still some classical correlation and some entanglement: indeed we find

$$\rho_c \left(\frac{1}{2}, \frac{1}{2} \right) = \rho_c \left(-\frac{1}{2}, -\frac{1}{2} \right) = \frac{3}{2}, \quad \rho_c \left(\frac{1}{2}, -\frac{1}{2} \right) = \rho_c \left(-\frac{1}{2}, \frac{1}{2} \right) = \frac{1}{2},$$

while

$$\rho_e \left(\frac{1}{2}, \frac{1}{2} \right) = \rho_e \left(-\frac{1}{2}, -\frac{1}{2} \right) = \frac{2}{3}, \quad \rho_e \left(\frac{1}{2}, -\frac{1}{2} \right) = \rho_e \left(-\frac{1}{2}, \frac{1}{2} \right) = 2,$$

in agreement with the product rule $\rho_c \cdot \rho_e = \rho_t$. A classical correlation and an entanglement survive at the mixture of Eq. (18) even if $a = b$, though in absence of a total correlation. This example shows a new and unexpected effect: even if a state shows no total correlation at all, one can find both classical and quantum correlations that compensate each other.

APPENDIX

We refer to the two-qubit example of Section 3, and consider the observables A_1, A_2 there defined representing the z-component of the spin of the two subsystems. Again $J(A_1, A_2)$ denotes their joint observable and $E^{J(A_1, A_2)}$ is the corresponding PV measure. We are going to show that there is no entanglement between A_1 and A_2 at any bipartite separable mixed state represented by the density

operator

$$D = \sum_i w_i \mathcal{P}_i \otimes \mathcal{Q}_i,$$

where $\mathcal{P}_i$ and $\mathcal{Q}_i$ are one-dimensional projectors of C^2, with i ranging over the positive integers (actually what we are going to show easily generalizes to integrals of product pure states with respect to arbitrary probability measures over pure states of the two-qubit system).

In order to calculate the entanglement function we have to compare the two measures $J(A_1, A_2)(D)$ and $\sum_i w_i A_1(\mathcal{P}_i \otimes \mathcal{Q}_i) \boxtimes A_2(\mathcal{P}_i \otimes \mathcal{Q}_i)$.

Looking at the explicit expression of $E^{J(A_1, A_2)}$ given in Section 3, and recalling that ψ_+, ψ_- denote the orthonormal vectors of C^2 representing the spin-up and spin-down states along the z-axis, we obtain for $J(A_1, A_2)(D)$ the explicit form

$$\sum_i w_i(\psi_+, \mathcal{P}_i \psi_+)(\psi_+, \mathcal{Q}_i \psi_+) \eta_{(\frac{1}{2}, \frac{1}{2})}$$

$$+ \sum_i w_i(\psi_-, \mathcal{P}_i \psi_-)(\psi_-, \mathcal{Q}_i \psi_-) \eta_{(-\frac{1}{2}, -\frac{1}{2})}$$

$$+ \sum_i w_i(\psi_+, \mathcal{P}_i \psi_+)(\psi_-, \mathcal{Q}_i \psi_-) \eta_{(\frac{1}{2}, -\frac{1}{2})}$$

$$+ \sum_i w_i(\psi_-, \mathcal{P}_i \psi_-)(\psi_+, \mathcal{Q}_i \psi_+) \eta_{(-\frac{1}{2}, \frac{1}{2})}$$

In fact, looking for instance at the first term of the above expression, we have just to refer to Eq. (1) and recall that at the point $\{\frac{1}{2}, \frac{1}{2}\}$ the PV measure $E^{J(A_1, A_2)}$ takes the value P_{++}, i.e., the projector onto the state $\psi_+ \otimes \psi_+$, so that the value of $J(A_1, A_2)(D)$ at that point becomes

$$\sum_i w_i \operatorname{Tr}(P_{++} \cdot \mathcal{P}_i \otimes \mathcal{Q}_i) = \sum_i w_i(\psi_+ \otimes \psi_+, \mathcal{P}_i \otimes \mathcal{Q}_i \psi_+ \otimes \psi_+)$$

(with similar remarks applying for the other terms).

On the other hand we have

$$A_1(\mathcal{P}_i \otimes \mathcal{Q}_i) = (\psi_+, \mathcal{P}_i \psi_+) \eta_{\frac{1}{2}} + (\psi_-, \mathcal{P}_i \psi_-) \eta_{-\frac{1}{2}}$$

and

$$A_2(\mathcal{P}_i \otimes \mathcal{Q}_i) = (\psi_+, \mathcal{Q}_i \psi_+) \eta_{\frac{1}{2}} + (\psi_-, \mathcal{Q}_i \psi_-) \eta_{-\frac{1}{2}}$$

as one sees by noticing, for instance, that at the point $\{\frac{1}{2}\}$ the PV measure E^{A_1} takes the value $P_+ \otimes I$ (see Section 3), so that at this point the value of $A_1(\mathcal{P}_i \otimes \mathcal{Q}_i)$ becomes (see Eq. (1)) $\operatorname{Tr}(P_+ \otimes I \cdot \mathcal{P}_i \otimes \mathcal{Q}_i) = \operatorname{Tr}(P_+ \mathcal{P}_i) = (\psi_+, \mathcal{P}_i \psi_+)$; and similarly for the other terms. Hence the mixture $\sum_i w_i A_1(\mathcal{P}_i \otimes \mathcal{Q}_i) \boxtimes A_2(\mathcal{P}_i \otimes \mathcal{Q}_i)$ of the product measures is immediately seen to reproduce exactly the measure

$J(A_1, A_2)(D)$, so that we conclude that the entanglement density function ρ_e is indeed the constant unit function.

As expected, there can be classical correlations between the two observables A_1, A_2 at the state D. In fact we have

$$A_1(D) = \sum_i w_i(\psi_+, \mathcal{P}_i\psi_+)\eta_{\frac{1}{2}} + \sum_i w_i(\psi_-, \mathcal{P}_i\psi_-)\eta_{-\frac{1}{2}}$$

and

$$A_2(D) = \sum_i w_i(\psi_+, \mathcal{Q}_i\psi_+)\eta_{\frac{1}{2}} + \sum_i w_i(\psi_-, \mathcal{Q}_i\psi_-)\eta_{-\frac{1}{2}},$$

so that the product measure takes the form

$$A_1(D) \boxtimes A_2(D) = \left(\sum w_i(\psi_+, \mathcal{P}_i\psi_+)\right)\left(\sum_i w_i(\psi_+, \mathcal{Q}_i\psi_+)\right)\eta_{(\frac{1}{2}, \frac{1}{2})}$$

$$+ \left(\sum w_i(\psi_-, \mathcal{P}_i\psi_-)\right)\left(\sum_i w_i(\psi_-, \mathcal{Q}_i\psi_-)\right)\eta_{(-\frac{1}{2}, -\frac{1}{2})}$$

$$+ \left(\sum w_i(\psi_+, \mathcal{P}_i\psi_+)\right)\left(\sum_i w_i(\psi_-, \mathcal{Q}_i\psi_-)\right)\eta_{(\frac{1}{2}, -\frac{1}{2})}$$

$$+ \left(\sum w_i(\psi_-, \mathcal{P}_i\psi_-)\right)\left(\sum_i w_i(\psi_+, \mathcal{Q}_i\psi_+)\right)\eta_{(-\frac{1}{2}, \frac{1}{2})}.$$

Clearly this measure does not coincide, in general, with $J(A_1, A_2)(D)$ so that we get a nontrivial density function ρ_c: the considered spin observables A_1, A_2 can exhibit classical correlations at D.

To exemplify the above results, let us consider the particular case

$$D = w P_+ \otimes P_+ + (1 - w)P_x \otimes P_x,$$

where P_+ is the the projector on the spin-up state ψ_+ along the z-axis while P_x is the projector on the eigenstate of the x-component of the spin corresponding to the eigenvalue $+\frac{1}{2}$.

Noticing that $(\psi_+, P_x\psi_+) = (\psi_-, P_x\psi_-) = \frac{1}{2}$ we see, by inspection of the previous formulas, that both the measures $J(A_1, A_2)(D)$ and $w A_1(P_+ \otimes P_+) \boxtimes A_2(P_+ \otimes P_+) + (1 - w)A_1(P_x \otimes P_x) \boxtimes A_2(P_x \times P_x)$ now take the form

$$\frac{1 + 3w}{4}\eta_{(\frac{1}{2}, \frac{1}{2})} + \frac{1 - w}{4}\eta_{(-\frac{1}{2}, -\frac{1}{2})} + \frac{1 - w}{4}\eta_{(\frac{1}{2}, -\frac{1}{2})} + \frac{1 - w}{4}\eta_{(-\frac{1}{2}, \frac{1}{2})},$$

leaving no room for entanglement.

On the other hand the product measure $A_1(D) \boxtimes A_2(D)$ now reads

$$\frac{(1+w)^2}{4}\eta_{(\frac{1}{2},\frac{1}{2})} + \frac{(1-w)^2}{4}\eta_{(-\frac{1}{2},-\frac{1}{2})} + \frac{1-w^2}{4}\eta_{(\frac{1}{2},-\frac{1}{2})} + \frac{1-w^2}{4}\eta_{(-\frac{1}{2},\frac{1}{2})},$$

so that the classical correlation function becomes, for $w \neq 1$,

$$\rho_c\left(\frac{1}{2},\frac{1}{2}\right) = \frac{1+3w}{(1+w)^2}, \quad \rho_c\left(-\frac{1}{2},-\frac{1}{2}\right) = \frac{1}{1-w},$$

$$\rho_c\left(\frac{1}{2},-\frac{1}{2}\right) = \frac{1}{1+w}, \quad \rho_c\left(-\frac{1}{2},\frac{1}{2}\right) = \frac{1}{1+w},$$

while for $w = 1$ both $J(A_1, A_2)(D)$ and $A_1(D) \boxtimes A_2(D)$ become concentrated at the point $\{\frac{1}{2}, \frac{1}{2}\}$.

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REFERENCES

Bauer, H. (1981). *Probability Theory and Elements of Measure Theory*, Academic Press, London.

Beltrametti, E. G. and Bugajski, S. (1995). A classical extension of quantum mechanics, *Journal of Physics A: Mathematical and General* **28**, 3329.

Beltrametti, E. G. and Bugajski, S. (1996). The Bell Phenomenon in classical frameworks, *Journal of Physics A: Mathematical and General* **29**, 247.

Beltrametti, E. G. and Bugajski, S. (2002). *Correlations and Entanglement in Probability Theory.* (arXiv:quant-ph/0211083)

Bennet, C. H., Di Vincenzo, D. P., Smolin, J. A., and Wootters, W. K. (1996). Mixed state entanglement and quantum error correction, *Physical Review A* **54**, 3824.

Billingsley, P. (1979). *Probability and Measure*, Wiley, New York.

Bugajski, S. (1996). Fundamentals of fuzzy probability theory, *International Journal of Theoretical Physics* **35**, 2229

Bugajski, S. (2001). Statistical maps: I. Basic properties, II. Operational random variables and the Bell phenomenon, *Mathematica Slovaca* **51**, 321, 343.

Busch, P., Grabowski, M., and Lahti, P. J. (1995). *Operational Quantum Physics*, Lecture Notes in Physics, m 31, 2nd edn, Springer, Berlin.

Busch, P., Lahti, P. J., and Mittelstaedt, P. (1996). *The Quantum Theory of Measurement*, Lecture Notes in Physics, m 2, 2nd edn. Springer, Berlin.

Henderson, L. and V. Vedral, (2001). *Classical, Quantum, and Total Correlations.* (arXiv:quant-ph/0105028)

Horodecki, M., Horodecki, P., and Horodecki, R. (2001). "Mixed-state entanglement and quantum communication." In *Quantum Information: An Introduction to Basic Theoretical Concepts and Experiments, C. Alber et al.,* eds., Springer, Berlin.

Keyl, M. (2002). Fundamentals of quantum information theory, *Physics Reports* **369**, 431.

Majewski, W. A. (2002). On Entanglement of States and Quantum Correlations. (arXiv:math-ph/0202030)